Lecture 10 **2012/12/03**

Outline

1. **Motivation**
2. **Stable Manifold Theorem**
3. **Summary**

1. Motivation

- The stable manifold theorem is one of the most important results in the local qualitative theory of ODE. It is similar to the linearization with different terminology: analytic way and geometric way.

2. Stable Manifold Theorem

1) The Linearized System

Consider the autonomous system

$$x' = f(x), \tag{10.1}$$

where $f(0)=0$ and $f \in C^1(B_r(0))$. The linearized system is given by

$$x' = Ax, \tag{10.2}$$

where $A = Df(0)$. Let $x=0$ be hyperbolic for both (10.1) and (10.2).

Suppose that A in (10.2) has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then $R^n = E^s \oplus E^u$ with $\dim E^s = k$ and $\dim E^u = n-k$. This situation can be essentially extended to (10.1) near any hyperbolic equilibrium.

2) Statement of Stable Manifold Theorem

Consider

$$x' = Ax + g(x), \tag{10.3}$$

where $g(x)$ is C^1 in U containing the origin, satisfying the basic condition as follows. $g(0)=0$ and $g'(0)=0$. Moreover, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x\|<\delta,\ \|y\|<\delta \quad \Rightarrow \quad \|g(x)-g(y)\| \le \varepsilon \|x-y\|. \tag{10.4}$$

Theorem 10.1 (Local Stable Manifold Theorem) Suppose that A in (10.3) has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Let $g(x)$ satisfy the basic condition. Then, there exist

1. an k-dimensional stable manifold $S = W^s(0)$ of class C^1 for (10.3) with $\dim W^s(0) = \dim E^s$, tangent to the stable subspace E^s at $x=0$, which is invariant under the flow φ_t of (10.3) and φ_t that starts on $W^s(0)$ is exponentially decay as $t \to +\infty$;
2. an $n-k$-dimensional unstable manifold $U = W^u(0)$ of class C^1 for (10.3) with $\dim W^u(0) = \dim E^u$, tangent to the unstable subspace E^u at $x=0$, which is invariant under the flow φ_t of (10.3) and φ_t that starts on $W^u(0)$ is exponentially decay as $t \to -\infty$.

3) An illustrative Example

Consider the following nonlinear system

$$x' = f(x),$$

where $f(x) = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}$. $x=0$ is only equilibrium. The linearized system is given by

$$x' = Df(0)x,$$

where $A = Df(0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Obviously, the stable subspace E^s is x_1 - x_2 plane and unstable subspace E^u is x_3 -axis. The solution with $x(0) = c = (c_1, c_2, c_3)^T$ is easily solved by

$$x_1(t) = c_1 e^{-t}; \quad x_2(t) = c_2 e^{-t} + c_1^2 (e^{-t} - e^{-2t}); \quad x_3(t) = c_3 e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}).$$

Clearly, $\lim_{t \to +\infty} \varphi_t(c) = 0 \iff c_3 + \frac{c_1^2}{3} = 0$. Thus,

$$W^s = \{c \in R^3 \mid c_3 + \frac{c_1^2}{3} = 0\}.$$

Similarly, $\lim_{t \to -\infty} \varphi_t(c) = 0 \iff c_1 = c_2 = 0$. Then,

$$W^u = \{c \in R^3 \mid c_1 = c_2 = 0\}.$$

The surface of the stable manifold S for this system is shown in Fig. 10.1, where S is tangent to E^s at $x = 0$ and the surface of the unstable manifold U is identical to E^u.

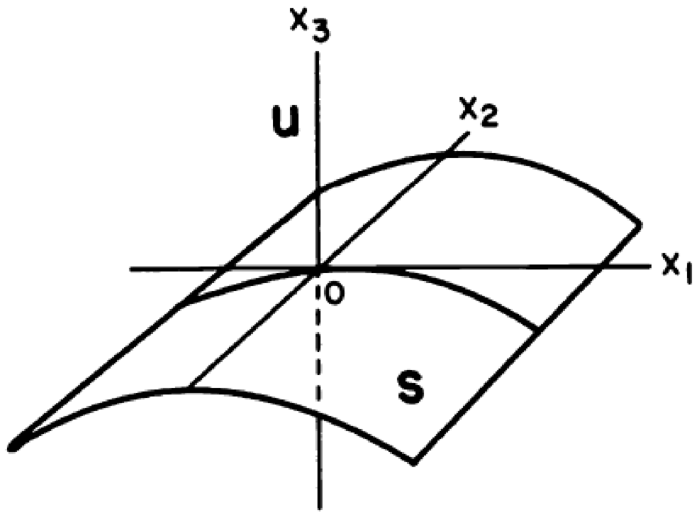

Fig. 10.1

3) Proof of (Local) Stable Manifold Theorem

Proof. Since A has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part, there exists an $n \times n$ invertible matrix C s.t.

$$B = C^{-1}AC = \begin{pmatrix} P & O \\ O & Q \end{pmatrix},$$

where the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ of $k \times k$ matrix P have negative real part and the eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_n$ of $(n-k)\times(n-k)$ matrix Q have positive real part. Under the transformation $x = C\,y$, (10.3) becomes

$$y' = B\,y + h(y), \tag{10.5}$$

where $h(y) = C^{-1}g(Cy)$ still satisfy the basic condition. For simplicity, we still use the same notation of ε and δ used in (10.4).

It will be first shown in the proof that there exist $n-k$ differential functions

$$y_j = \psi_j(y_1, y_2, \cdots, y_k), \quad j = k+1, \cdots, n,$$

that define a k- dimensional differential manifold $\tilde{S}$ for (10.5). Then, the local stable manifold S for (10.3) is obtained by $x = Cy$.

Let

$$U(t) = \begin{pmatrix} e^{Pt} & O \\ O & O \end{pmatrix} \text{ and } V(t) = \begin{pmatrix} O & O \\ O & e^{Qt} \end{pmatrix}.$$

Then,

$$e^{Bt} = U(t) + V(t)$$

and

$$U'(t) = PU(t) = BU(t), \quad V'(t) = QV(t) = BV(t).$$

Then, we choose $\alpha > 0$ sufficiently small s.t. $\max_{j=1,\cdots,k} \operatorname{Re}\lambda_j < -\alpha < 0$. For such a $\alpha > 0$ we can choose $K > 0$ sufficiently large and $\sigma > 0$ sufficiently small such that

$$\|U(t)\| \le K e^{-(\alpha+\sigma)t} \text{ for all } t \ge 0; \quad \|V(t)\| \le K e^{\sigma t} \text{ for all } t \le 0.$$

Next consider the integral equation

$$u(t, a) = U(t)a + \int_0^t U(t-s)h(u(s, a))\,ds - \int_t^\infty V(t-s)h(u(s, a))\,ds. \qquad (10.6)$$

If $u(t, a)$ is a continuous solution of (10.6), then, differentiating (10.6) on both sides yields

$$\begin{aligned} u'(t, a) &= U'(t)a + \int_0^t U'(t-s)h(u(s, a))\,ds - \int_t^\infty V'(t-s)h(u(s, a))\,ds \\ &\quad + [U(0) + V(0)]\,h(u(t, a)) \\ &= B\{U(t)a + \int_0^t U(t-s)h(u(s, a))\,ds - \int_t^\infty V(t-s)h(u(s, a))\,ds\} + h(u(t, a)) \\ &= Bu(t, a) + h(u(t, a)). \end{aligned}$$

Then, $u(t, a)$ is a smooth solution of (10.5). We solve (10.6) by a successive approximation. Before solving (10.6), we assume $\varepsilon \le \dfrac{\sigma}{4K}$ and $\|a\| \le \dfrac{\delta}{2K}$ because it is for the local. Let

$$u^{(0)}(t,a)=0;$$

$$u^{(j+1)}(t,a)=U(t)a+\int_0^t U(t-s)h(u^{(j)}(s,a))\,ds-\int_t^\infty V(t-s)h(u^{(j)}(s,a))\,ds.$$

We will establish two estimates by induction as follows.

$$\|u^{(j)}(t,a)\|\le 2K\|a\|e^{-\alpha t},\ \text{ for }\ t\ge 0. \tag{10.7}$$

$$\|u^{(j)}(t,a)-u^{(j-1)}(t,a)\|\le \frac{K\|a\|e^{-\alpha t}}{2^{j-1}},\ \text{ for }\ t\ge 0. \tag{10.8}$$

To show (10.7), it holds for $j=0$ at first. Assume that (10.7) holds for $j=m$, i.e.

$$\|u^{(m)}(t,a)\|\le 2K\|a\|e^{-\alpha t},\ \text{for}\ t\ge 0.$$

Then, $\|u^{(m)}(t,a)\|\le 2K\|a\|\le\delta$. Notice that for $V(t-s)$, $t-s\le 0$ as $s\ge t$. It follows that

$$\|u^{(m+1)}(t,a)\|\le\|U(t)\|\|a\|+\varepsilon\int_0^t\|U(t-s)\|\|u^{(m)}(s,a)\|ds+\varepsilon\int_t^\infty\|V(t-s)\|\|u^{(m)}(s,a)\|ds$$

$$\le Ke^{-(\alpha+\sigma)t}\|a\|+2\varepsilon K^2\|a\|\int_0^t e^{-(\alpha+\sigma)(t-s)}e^{-\alpha s}\,ds+2\varepsilon K^2\|a\|\int_t^\infty e^{\sigma(t-s)}e^{-\alpha s}ds$$

$$\le Ke^{-(\alpha+\sigma)t}\|a\|+2\varepsilon K^2\|a\|\int_0^t e^{-(\alpha+\sigma)(t-s)}e^{-\alpha s}\,ds+2\varepsilon K^2\|a\|\int_t^\infty e^{\sigma(t-s)}e^{-\alpha s}ds$$

$$\le Ke^{-(\alpha+\sigma)t}\|a\|+2\varepsilon K^2\|a\|e^{-\alpha t}\frac{1}{\sigma}+2\varepsilon K^2\|a\|e^{-\alpha t}\frac{1}{\sigma+\alpha}$$

$$\le K\|a\|(e^{-(\alpha+\sigma)t}+e^{-\alpha t})\le 2K\|a\|e^{-\alpha t},\ \text{for}\ t\ge 0.$$

By induction, we have (10.7) for all j.

To show (10.8), we show that (10.8) holds for $j=1$ at first. Since

$$\|u^{(1)}(t,a)-u^{(0)}(t,a)\|=\|u^{(1)}(t,a)\|=\|U(t)a\|\le\|U(t)\|\|a\|$$

$$\le K\|a\|e^{-(\alpha+\sigma)t}\le K\|a\|e^{-\alpha t},\ \text{for}\ t\ge 0,$$

then (10.8) holds for $j=1$. We assume that (10.8) holds for $j=m$. Notice that since $\|u^{(j)}(t,a)\|\le 2K\|a\|\le\delta$ for all j and $t\ge 0$ by (10.7), we have

$$\|h(u^{(m)}(t,a))-h(u^{(m-1)}(t,a))\|\le\varepsilon\|u^{(m)}(t,a)-u^{(m-1)}(t,a)\|,\ \ t\ge 0.$$

Then,

$$\| u^{(m+1)}(t,a)) - h(u^{(m)}(t,a)) \| \le \int_0^t \| U(t-s) \| \cdot \| h(u^{(m)}(s,a)) - h(u^{(m-1)}(s,a)) \| ds$$

$$+ \int_t^\infty \| V(t-s) \| \cdot \| h(u^{(m)}(s,a)) - h(u^{(m-1)}(s,a)) \| ds$$

$$\le \varepsilon K \int_o^t e^{-(\alpha+\sigma)(t-s)} \cdot \frac{K \| a \| e^{-\alpha s}}{2^{m-1}} ds + \varepsilon K \int_t^\infty e^{-\sigma(t-s)} \cdot \frac{K \| a \| e^{-\alpha s}}{2^{m-1}} ds$$

$$\le \frac{\varepsilon K^2 \| a \|}{2^{m-1}} \{ e^{-(\alpha+\sigma)t} \int_o^t e^{\sigma s} ds + e^{-\sigma t} \int_t^\infty e^{-(\sigma+\alpha)s} ds \}$$

$$\le \frac{\varepsilon K^2 \| a \|}{2^{m-1}} (\frac{1}{\sigma} + \frac{1}{\sigma+\alpha}) e^{-\alpha t} \le \frac{\varepsilon K^2 \| a \|}{2^{m-2}} \cdot \frac{1}{\sigma} e^{-\alpha t} \le \frac{K \| a \|}{2^m} e^{-\alpha t}, \quad t \ge 0.$$

By induction, we have (10.8) for all j and $t \ge 0$. Thus, for $n > m \ge N$ and $t \ge 0$,

$$\| u^{(n)}(t,a) - u^{(m)}(t,a) \| = \sum_{j=m}^{n} \| u^{(j+1)}(t,a) - u^{(j)}(t,a) \| \le \sum_{j=N}^{\infty} \| u^{(j+1)}(t,a) - u^{(j)}(t,a) \|$$

$$\le \sum_{j=N}^{\infty} \frac{K \| a \| e^{-\alpha t}}{2^j} \le K \| a \| \sum_{j=N}^{\infty} \frac{1}{2^j} = \frac{K \| a \|}{2^{N-1}} \to 0$$

as $N \to \infty$ uniformly for all $t \ge 0$. Therefore, $\{u^{(j)}(t,a)\}$ is a uniform Cauchy sequence for $t \ge 0$. Then, $\lim_{j \to \infty} u^{(j)}(t,a) = u(t,a)$, which is continuous for $t \ge 0$. Moreover, by (10.7), we have

$$\| u(t,a) \| \le 2K \| a \| e^{-\alpha t} \quad \text{for } t \ge 0, \text{ as } \| a \| \le \frac{\delta}{2K}. \qquad (10.9)$$

Taking limit on both sides of (10.8), $u(t,a)$ is the solution of (10.6).

Look at (10.8) and notice that $U(t)a$, we know that the only previous k components of a can determine $u(t,a)$. Hence the last $n-k$ components of a may and will take all 0. That is, $u(t,a) = (u_1(t,a), u_2(t,a), \cdots, u_n(t,a))^T$ satisfy the initial conditions

$$u_j(0,a) = a_j, \quad j = 1, 2, \cdots, k;$$

and based on (10.6) we have

$$u_j(0,a) = -\int_0^\infty V(-s) h(u(s, a_1, a_2, \cdots a_k, 0, \cdots, 0)) ds, \quad j = k+1, k+2, \cdots, n.$$

Now we are in the position to define the manifold $\Psi = (\psi_{k+1}, \psi_{k+2}, \cdots, \psi_n)$ by

$$\psi_j(a_1, a_2, \cdots, a_k) = u_j(0, a_1, a_2, \cdots, a_k, 0, \cdots, 0)\,. \tag{10.10}$$

Then, the initial values $y_j = u_j(0, a_1, a_2, \cdots, a_k, 0, \cdots, 0)$ satisfy

$$y_j = \psi_j(y_1, y_2, \cdots, y_k) \quad \text{for} \quad j = k+1, k+2, \cdots, n\,.$$

According to (10.10), these equations define a manifold $\tilde{S}$ in the domain of $\sqrt{y_1^2 + y_2^2 + \cdots + y_k^2} < \dfrac{\delta}{2K}$. $\tilde{S}$ is also differentiable (Similar to show that solutions are differentiable w.r.t. initial conditions. Omitted). It is locally defined.

Show that $\tilde{S}$ is invariant. If $y(t)$ is a solution of (10.5) with $y(0) \in \tilde{S}$, i.e. $y(0) = u(0, a)$, then, $y(t) = u(t, a)$ by uniqueness. Therefore $y(t) \in \tilde{S}$ and by (10.9) $\lim_{t \to +\infty} y(t) = 0$. Therefore, $\tilde{S}$ is the stable invariant manifold.

Show that $\tilde{S}$ is unique. If $y(t)$ is a solution of (10.5) with $y(0) \notin \tilde{S}$, then, $\lim_{t \to +\infty} y(t) \neq 0$. Show it by contradiction. Suppose that $\| y(t) \| \le \delta$ for all $t \ge 0$. Solving (10.5), we have

$$\begin{aligned}
y(t) &= e^{Bt} y(0) + \int_0^t e^{B(t-s)} h(y(s))\, ds \\
&= (U(t) + V(t)) y(0) + \int_0^t (U(t-s) + V(t-s)) h(y(s))\, ds \\
&= (U(t) + V(t)) y(0) + \int_0^t U(t-s) h(y(s))\, ds + \int_0^t V(t-s)) h(y(s))\, ds \\
&= (U(t) + V(t)) y(0) + \int_0^t U(t-s) h(y(s))\, ds - \int_t^\infty V(t-s) h(y(s))\, ds \\
&\quad + \int_0^\infty V(t-s) h(y(s))\, ds \\
&= U(t) y(0) + V(t) c + \int_0^t U(t-s) h(y(s))\, ds - \int_t^\infty V(t-s) h(y(s))\, ds\,,
\end{aligned}$$

where $c = y(0) + \int_0^\infty V(-s) h(y(s))\, ds$ is finite because the infinite integral converges. In the expression of $y(t)$, $V(t)$ is unbounded because Q has eigenvalues with all positive real parts unless $c = 0$. But if $c = 0$, then $y(0) = -\int_0^\infty V(-s) h(y(s))\, ds \in \tilde{S}$. This is a contradiction. The contradiction shows that $\tilde{S}$ is locally unique.

Show that $\tilde{S}$ is tangent to the stable subspace

$$E^s = \{y \in R^n \mid y_{k+1} = y_{k+2} = \cdots = y_n = 0\}$$

of (10.5) at $y = 0$. Notice that $u_j(t, 0)|_{t=0} = 0$ and $\| u(t,a) \| \le 2K \| a \| e^{-\alpha t}$, and

$$\| u_j(0, a) \| \le \int_0^\infty \| V(-s) \| \| h(u(s,a)) \| ds \le \varepsilon \int_0^\infty \| V(-s) \| \| u(s,a) \| ds$$

$$\le \varepsilon\, 2K^2 \| a \| \int_0^\infty e^{-(\sigma+\alpha)s}\, ds = \varepsilon\, 2K^2 \| a \| \frac{1}{\sigma+\alpha} \le 2\varepsilon K^2 \| a \| \frac{1}{\sigma},$$

we have

$$\frac{\partial \psi_j}{\partial y_i}(0) = \lim_{y_i \to 0} \frac{\psi_j(0,\cdots,0,y_i,0,\cdots,0) - \psi_j(0,\cdots,0)}{y_i} = \lim_{y_i \to 0} \frac{\psi_j(0,\cdots,0,y_i,0,\cdots,0)}{y_i};$$

$$\psi_j(0,\cdots,0,y_i,0,\cdots,0) = u_j(t,0,\cdots,0,y_i,0,\cdots,0,y_{k+1},\cdots,y_{k+1})\Big|_{\substack{t=0 \\ y_{k+1}=\cdots=y_n=0}};$$

$$\left\| \frac{\psi_j(0,\cdots,0,y_i,0,\cdots,0)}{y_i} \right\| \le \frac{2\varepsilon K^2}{\sigma},$$

where $\varepsilon > 0$ can be made arbitrarily small by $\| a \| << 1$, which can be done by letting $\| y_i \| << 1$. Therefore, $\dfrac{\partial \psi_j}{\partial y_i}(0) = 0$. This shows that $\tilde{S}$ is tangent to E^s at $y = 0$. That is, S is tangent to E^s at $x = 0$.

The existence of the unstable manifold $\tilde{U}$ of (10.5) is shown exactly the same way by reversing $t \to -t$. By considering

$$y' = -By - h(y), \tag{10.11}$$

the stable manifold for (10.11) is the unstable manifold of (10.5). This concludes the proof. □

Remark 10.1 If $g(x) \in C^r$, $r \ge 1$, then the stable and unstable manifolds S and U are also class C^r.

4) An Example for Construction of Successive Approximations

Consider the nonlinear system

$$\begin{cases} x_1' = -x_1 - x_2^2 \\ x_2' = x_2 - x_1^2 \end{cases}.$$

For this system, we have

$$A = B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g(x) = h(x) = \begin{pmatrix} -x_1^2 \\ x_1^2 \end{pmatrix};$$

$$U(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^t \end{pmatrix}, \quad e^{Bt} = U(t) + V(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix};$$

$$a = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, \quad n = 2 \quad \text{and} \quad k = 1.$$

The integral equation for $u(t,a)$ is given by

$$u(t,a) = \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-(t-s)} u_2^2(s,a) \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{t-s} u_1^2(s,a) \end{pmatrix} ds.$$

We find

$$u^{(0)}(t,a) = 0; \quad u^{(1)}(t,a) = \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix};$$

$$u^{(2)}(t,a) = \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix} - \int_t^\infty \begin{pmatrix} 0 \\ e^{t-s} e^{-2s} a_1^2 \end{pmatrix} ds = \begin{pmatrix} e^{-t} a_1 \\ -\dfrac{e^{-2t}}{3} a_1^2 \end{pmatrix};$$

$$u^{(3)}(t,a) = \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix} - \frac{1}{9} \int_0^t \begin{pmatrix} e^{-(t-s)} e^{-4s} a_1^4 \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{t-s} e^{-2s} a_1^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t} a_1 + \dfrac{1}{27}(e^{-4t} - e^{-t}) a_1^4 \\ -\dfrac{1}{3} e^{-2t} a_1^2 \end{pmatrix}.$$

It can be shown that $u^{(4)}(t,a) - u^{(3)}(t,a) = o(a_1^5)$ and therefore $\psi_2(a_1) = u_2(0, a_1, 0)$ is approximated by

$$\psi_2(a_1) = u_2(0, a_1, 0) = -\frac{1}{3} a_1^2 + o(a_1^5)$$

as $a_1 \to 0$. Hence, the local stable manifold S is approximated by

$$S : x_2 = -\frac{1}{3} x_1^2 + o(x_1^5),$$

as $x_1 \to 0$. The local unstable manifold U is approximated by applying exactly the same procedure to the above system with $t \to -t$, x_1 and x_2 interchanged as follows.

$$U: x_1 = -\frac{1}{3}x_2^2 + o(x_2^5),$$

as $x_2 \to 0$. These approximations for S and U near the origin, E^s and E^u for $x' = Ax$ are shown in Fig. 10.2.

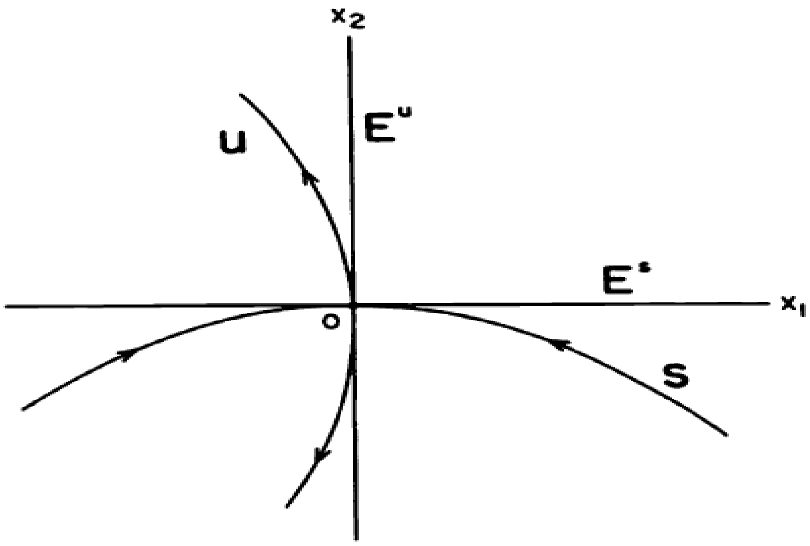

Fig. 10.2

5) Global Stable and Unstable Manifolds

Definition 10.1 Let φ_t be the flow of (10.3). The global stable and unstable manifolds of (10.3) at the origin are defined by

$$W^s(0) = \bigcup_{t \le 0} \varphi_t(S) \text{ and } W^u(0) = \bigcup_{t \ge 0} \varphi_t(U).$$

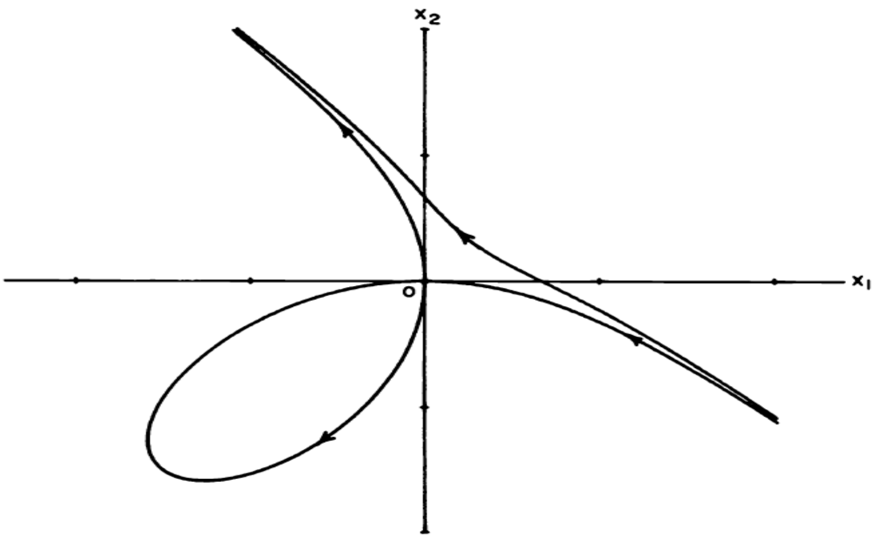

Fig. 10.3

Remark 10.2 For all $x \in W^s(0)$, $\lim_{t\to\infty} \varphi_t(x) = 0$ and for all $x \in W^u(0)$, $\lim_{t\to-\infty} \varphi_t(x) = 0$.

Remark 10.3 Fig. 10.3 shows some numerically computed solution curves for the example 4). The global stable and unstable manifolds $W^s(0)$ and $W^u(0)$ for the same example are shown in Fig.10.4.

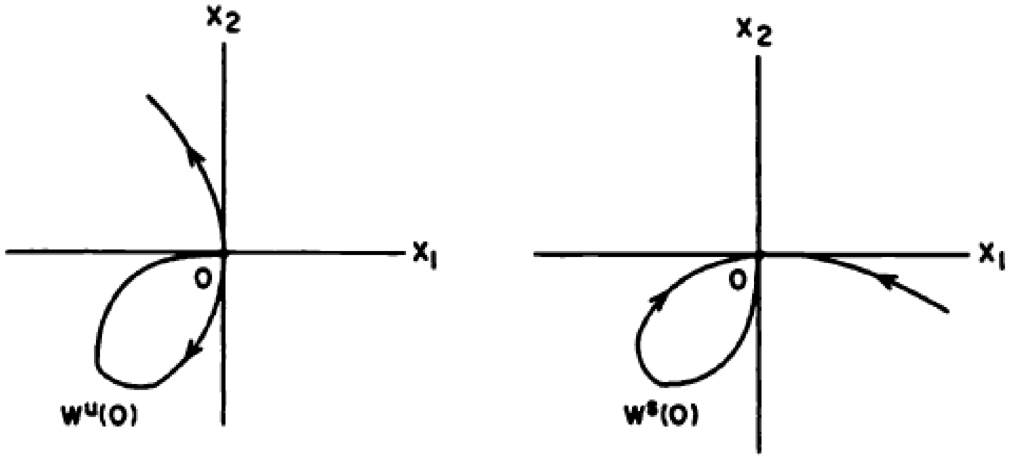

Fig. 10.4

3. Summary

- Although the stable manifold theorem and the linearization characterize that $x' = f(x)$ and $x' = Df(0)x$ have the same stability property near a hyperbolic equilibrium, the stable manifold theorem gives much more information on geometric structures.

- The stable manifold theorem uses a geometric way to characterize the local property near a hyperbolic equilibrium. The linearization uses an analytical way to characterize the local property near a hyperbolic equilibrium.

- Stable and unstable manifolds are both lower dimensional smooth surfaces in R^n. From Lyapunov stability, even the neighborhood of equilibrium is n dimensional domain. Therefore, if there exists an unstable manifold, it is definitely unstable in the sense of Lyapunov stability.

Homework Review today's lecture and understand the details.